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2001 J. Phys. A: Math. Gen. 34 4601

(http://iopscience.iop.org/0305-4470/34/22/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 4601–4614 www.iop.org/Journals/ja PII: S0305-4470(01)19489-8

Enhanced generation rate of the coherent entanglement photon pairs in parametrical down-conversion

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Received 24 November 2000, in final form 9 April 2001

Abstract

The effect of enhancing nonlinear generation of entangled photons in the process of interaction of the external coherent electromagnetic field with a nonlinear dispersive medium is studied in this paper. Taking into account the second- and third-order susceptibility tensors of the crystal, it is demonstrated that in the good cavity approximation the bistable behaviour of the two-photon generation coefficient as a function of intensity of the pump laser field is possible. This effect is stimulated by decreasing the detuning between the frequency of the cavity mode and pump frequency as a function of anharmonicity terms in polarization.

PACS numbers: 4250, 4120

1. Introduction

The problem of quantum fluctuations and the generation of the non-classical electromagnetic field (EMF) in two-photon and multi-photon processes has recently been the subject of a number of theoretical and experimental studies [1]. The entanglement phenomenon between idler and signal photons generated in the parametric down-conversion has been intensively studied in the last decade. For example, such effects as quantum interference [2] and non-locality [3] are possible thanks to the extremely short correlation time between two photons produced in the large band of parametrical down-conversion [4]. In one-dimension approximation the broad-band squeezed-vacuum EMF consists of pairs of entanglement photons which can coherently excite the dipole-forbidden transitions like coherent EMF [5]. The effects of coherent excitation arise in the problem of generating more powerful broad-band squeezed light in the parametrical down-conversion [6].

The aim of this paper is to study the process of generation of entanglement photon pairs in a nonlinear cavity which contains the second- and third-order nonlinearity driven by a strong external coherent laser field. Let us consider the situation when the parametric oscillator

0305-4470/01/224601+14$30.00

consists of a crystal in a double-resonant cavity with mirrors which almost completely reflect the subharmonic light and reflect the pump light badly. If the mirrors only reflect the subharmonic light one can adiabatically eliminate the operators of the pump light. Further, a large number of discrete modes for the subharmonic light inside the cavity which contains a nonlinear disperse medium is considered. If the double-cavity frequency of entanglement photons $\omega_{2k_0} = \omega_k + \omega_{2k_0-k}$ is off-resonance with the pump field ω_{p}, the detuning $\Delta_0 = \omega_{\mathrm{p}} - \omega_{2k_0}$ extinguishes the process of generation of entanglement photons in the cavity. In this situation, the third-order nonlinearity can diminish the detuning factor in the nonlinear dispersive medium as a function of intensity of the pump field. In this critical point a more powerful enhancement of the generation rate of entanglement photons is observed.

This effect differs from the traditional parametrical oscillator [7] due to the fact that the number of conjugate mode pairs k_i, $2k_0 - k_i$ $(i = 1, 2, \ldots, N)$ in which entanglement photon pairs are generated is larger and the cavity is bad for the pump field, so that we can eliminate the fluctuation part of the pump field. In studying the generation of entanglement photon pairs the third term in polarization decomposition is also taken into account, as is the influence of the detuning between the pump frequency and the double-frequency subharmonic field (Δ_0) on the generation rate.

Our attention is mainly focused on the dependence of the number of entanglement photon pairs as a function of the low and high pump field intensity.

A new master equation for the coupled subharmonic EMF with an external-driven coherent field is obtained. The coupled system obeys the $SU(1, 1)$ symmetry and a Casimir pseudovector operator for $SU(1, 1)$ algebra is conserved. Using the generalized P-representation of the Fokker–Planck equation for $SU(1, 1)$ symmetry the proposed master equation is obtained. In order to obtain the steady state solution of the master equation two methods are proposed. The first method is based on the stationary solution of the Fokker–Planck equation and the second represents the density matrix through antinormal products of creation and annihilation operators of $SU(1, 1)$ algebra. The analytical and numerical results show that these two methods are not equivalent, and that the theory of stationary solutions for quantum master equations needs more careful development. A similar problem was analysed and solved for the case of a two-level system interacting with a coherent external field [8]. It is well known that such a two-level system obeys $SU(2)$ symmetry. However, in recent years it has also been realized that the $SU(1, 1)$ group plays an important role in many problems in quantum optics [9–11].

2. Master equation for the subharmonic field

The Hamiltonian which describes the interaction of the EMF with the nonlinear dispersive medium in the cavity can be obtained, following the Collett and Gardiner treatment [12], as

$$H = H_e + H_i + H_{\mathrm{c}}. \tag{1}$$

Here

$$H_e = \hbar \int_0^\infty \mathrm{d}\omega\, \omega B_\omega^\dagger B_\omega \tag{2}$$

is the free Hamiltonian for the external field modes. B_ω and $B_\omega^\dagger$ are the annihilation and creation operators for the external field which satisfy the commutation relation $[B_\omega, B_{\omega'}^\dagger] = \delta(\omega - \omega')$:

$$H_i = \mathrm{i}\hbar \int_{-\infty}^{+\infty} \mathrm{d}\omega\, k(\omega)[B_\omega b^\dagger - B_\omega^\dagger b] \tag{3}$$

is the interaction Hamiltonian between the external coherent field with frequency ω_{p} and the cavity field which contains a limited number of discrete field modes in the energy interval $(0, \hbar\omega_{2k_0})$. b and $b^\dagger$ are the annihilation and creation operators for the intracavity field with frequency near the pump $(\Delta_0 \approx 0)$; $k(\omega)$ is coupling constant. We consider that the cavity is good for the subharmonic field $(k(\omega_{k_0}) \approx 0)$ and for the high-frequency field $\omega \approx \omega_{\mathrm{p}}$ the coupling constant is large.

In order to obtain the intracavity Hamiltonian H_{c}, let us expand the polarization of the nonlinear medium to third order in the EMF strength,

$$P_\alpha = \chi^{(1)}_{\alpha\beta} E_\beta + \chi^{(2)}_{\alpha\beta\gamma} E_\beta E_\gamma + \chi^{(3)}_{\alpha\beta\gamma\delta} E_\beta E_\gamma E_\delta. \tag{4}$$

Here $\chi^{(n)}$ is a $(n+1)$th-rank susceptibility tensor. After introducing this polarization in the density part of the interaction Hamiltonian $H_{\mathrm{int}} = \int (P_\alpha(\vec{E}), \mathrm{d}E_\alpha)$ one can obtain the following form of intracavity Hamiltonian [6]:

$$H_{\mathrm{c}} = \int \mathrm{d}^3\vec{r}\, \left(\frac{|\vec{B}|^2}{2\mu_0} + \frac{1}{2}\big(\epsilon_0 + \chi^{(1)}_{\alpha\beta}\big) E_\alpha E_\beta + \frac{1}{3}\chi^{(2)}_{\alpha\beta\gamma} E_\alpha E_\beta E_\gamma + \frac{1}{4}\chi^{(3)}_{\alpha\beta\gamma\delta} E_\alpha E_\beta E_\gamma E_\delta \right). \tag{5}$$

As the external laser field pump is only the cavity mode $2k_0$ one can express the EMF strength $\vec{E}$ inside the cavity through the strength intracavity pump field $\vec{E}_{\mathrm{p}}$ and the subharmonic mode components generated in the process of parametrical down-conversion $\vec{E}_{\mathrm{sh}}$:

$$\vec{E} = \vec{E}_{\mathrm{p}} + \vec{E}_{\mathrm{sh}}. \tag{6}$$

$\vec{E}_{\mathrm{p}}$ can be expressed by the annihilation (b) and creation $(b^\dagger)$ operators in the form

$$\vec{E}_{\mathrm{p}}(\vec{r}, t) = \mathrm{i}\sqrt{\frac{\bar{\omega}_{\mathrm{p}}}{2\epsilon_0}}\big(b\vec{u}(\vec{r})\exp(-\mathrm{i}\omega_{\mathrm{p}}t) - b^\dagger\vec{u}^*\exp(\mathrm{i}\omega_{\mathrm{p}}t)\big) \tag{7}$$

where

$$\vec{u}(\vec{r}) = \frac{\vec{e}_\lambda}{\sqrt{V}}\exp[\mathrm{i}(\vec{k}_{\mathrm{p}}, \vec{r})].$$

Here V is the quantization volume and $\vec{e}_\lambda$ is the polarization vector of the EMF. The electric-field operator for the cavity mode of subharmonic frequencies $\omega_k \leqslant \omega_{2k_0}$ can be written as

$$\vec{E}_{\mathrm{sh}}(\vec{r}, t) = \mathrm{i}\sqrt{\frac{\bar{\omega}_{\mathrm{p}}}{2\epsilon_0}}\sum_{k=0}^{2k_0}\big(a_k\vec{v}_k(\vec{r})\exp(-\mathrm{i}\omega_k t) - a_k^\dagger\vec{v}_k^*(\vec{r})\exp(\mathrm{i}\omega_k t)\big) \tag{8}$$

where

$$\vec{v}_k(\vec{r}) = \frac{\vec{e}_\lambda}{\sqrt{V}}\exp[\mathrm{i}(\vec{k}, \vec{r})]$$

and a_k and $a_k^\dagger$ are the annihilation and creation photon operators in the cavity mode k. Taking into account that the losses of pump EMF in the cavity are larger than those of the subharmonic EMF one can represent the intracavity Hamiltonian as

$$\begin{aligned} H_{\mathrm{c}} = \hbar\tilde{\omega}_{\mathrm{p}} b^\dagger b + \hbar\chi^0 {b^\dagger}^2 b^2 + \hbar\sum_{k=0}^{2k_0}\big(\tilde{\omega}_k + \chi^0_k b^\dagger b\big)a_k^\dagger a_k \\ +\hbar\sum_{k_1=0}^{2k_0}\sum_{k_2=0}^{2k_0}\big(\chi'_{k_1,k_2} a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2} a_{2k_0-k_2} + \chi''_{k_1,k_2} a_{k_1}^\dagger a_{k_1} a_{k_2}^\dagger a_{k_2}\big) \\ +\mathrm{i}\sum_{k=0}^{2k_0} g_k\big(b^\dagger a_k a_{2k_0-k} - b a_k^\dagger a_{2k_0-k}^\dagger\big). \end{aligned} \tag{9}$$

Here the coefficients

$$\tilde{\omega}_a = \frac{1}{2}\left(1 + \frac{\chi^{(1)}}{\epsilon_0}\right)\omega_a$$

where $a = (p, k)$,

$$\chi^0 = \frac{3}{8}\frac{\chi^{(3)}(\omega_\mathrm{p}, \omega_\mathrm{p})\hbar\omega_\mathrm{p}}{\epsilon_0^2}\omega_\mathrm{p} \qquad \chi_k^0 = \frac{3}{2}\frac{\chi^{(3)}(\omega_\mathrm{p}, \omega_k)\hbar\omega_\mathrm{p}}{\epsilon_0^2}\omega_k$$

$$\chi'_{k_1,k_2} = \frac{3}{8}\frac{\chi^{(3)}(\omega_{k_1}, \omega_{k_2})\hbar}{\epsilon_0^2}\sqrt{\omega_{k_1}\omega_{2k_0-k_1}\omega_{k_2}\omega_{2k_0-k_2}} \qquad \chi''_{k_1,k_2} = \frac{3}{8}\frac{\chi^{(3)}(\omega_{k_1}, \omega_{k_2})\hbar}{\epsilon_0^2}\omega_{k_1}\omega_{k_2}$$

and

$$g_k = \sqrt{\frac{\hbar}{2\epsilon_0}}\frac{\chi^{(2)}}{\epsilon_0}\hbar\sqrt{\omega_\mathrm{p}\omega_k\omega_{2k_0-k}}$$

is the constant of the interaction of the intracavity pump and subharmonic fields with the frequencies ω_p and ω_k respectively, obtained from the second-order polarization expansion (1).

Let us consider the operator $\mathrm{O}(t)$ which belongs to the subharmonic EMF. Taking into account the Hamiltonian (1) one can write the following Heisenberg equation for this mean value of the operator:

$$\begin{aligned}\frac{\mathrm{d}\langle \mathrm{O}(t)\rangle}{\mathrm{d}t} &= \mathrm{i}\sum_{k=0}^{2k_0}\hbar\left(\tilde{\omega}_k + \chi_k^0 b^\dagger b\right)\left\langle\left[a_k^\dagger a_k, \mathrm{O}(t)\right]\right\rangle \\ &\quad + \mathrm{i}\hbar\sum_{k_1=0}^{2k_0}\sum_{k_1=0}^{2k_0}\left\langle\left[\chi'_{k_1,k_2}a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2}a_{2k_0-k_2} + \chi'_{k_1,k_2}a_{k_1}^\dagger a_{k_1}a_{k_2}^\dagger a_{k_2}, \mathrm{O}(t)\right]\right\rangle \\ &\quad - \frac{1}{\hbar}\sum_{k=0}^{2k_0} g_k\left\langle\left[b^\dagger a_k a_{2k_0-k} - b a_k^\dagger a_{2k_0-k}^\dagger, \mathrm{O}(t)\right]\right\rangle.\end{aligned} \tag{10}$$

In this equation we must eliminate the cavity operators of higher frequency $b(t)$ and $b^\dagger(t)$. Using the system Hamiltonian one can obtain the following Heisenberg equation for these operators:

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\mathrm{i}\left(\tilde{\omega}_\mathrm{p} + 2\chi^0 b^\dagger b + \sum_{k=0}^{2k_0}\chi_k^0 a_k^\dagger a_k\right)b + \frac{1}{2}\sum_{k=0}^{2k_0} g_k a_k a_{2k_0-k} + \int_{-\infty}^{+\infty}\mathrm{d}\omega\, k(\omega)B_\omega. \tag{11}$$

As the external EMF is in the single-mode coherent field $|\mathrm{i}n\rangle = \prod_{\omega_i\neq\omega_\mathrm{p}}|0\rangle_{\omega_i}\exp(\beta B_{\omega_\mathrm{p}}^\dagger - \beta^* B_{\omega_\mathrm{p}})|0\rangle_{\omega_\mathrm{p}}$, the solution of the Heisenberg equation for the external EMF can be represented as

$$B_\omega(t) = B_\omega(0)\mathrm{e}^{-\mathrm{i}\omega t} - k(\omega)\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau}b(t-\tau) \tag{12}$$

where $B_\omega(0)$ is the free part of the EMF operator, which satisfies the identity $B_\omega(0)|\mathrm{i}n\rangle = \beta\delta_{\omega,\omega_\mathrm{p}}|\mathrm{i}n\rangle$. After substitution of this solution in Heisenberg equation (11) one can represent the solution of operator b_{2k_0} in the following form:

$$\begin{aligned}b(t) &= b(0)\mathrm{e}^{-\mathrm{i}\omega_{2k_0}t} + \int_0^t \mathrm{d}\tau_1\, T\exp\left(-\mathrm{i}\int_{\tau_1}^t(\hat{\omega}(\tau) - \mathrm{i}\Gamma)\,\mathrm{d}\tau\right) \\ &\quad \times\left(\int_{-\infty}^{+\infty}k(\omega)B_\omega(0)\mathrm{e}^{-\mathrm{i}\omega\tau_1}\,\mathrm{d}\omega + \frac{1}{\hbar}\sum_{k=0}^{2k_0}g_k a_k(\tau_1)a_{2k_0-k}(\tau_1)\right).\end{aligned} \tag{13}$$

Here T represents the chronological product of the operators and $\Gamma = \pi k^2(\omega_{2k_0})$ is the cavity loss at frequency ω_{2k_0}. The coupling between cavity mode $2k_0$ and the external EMF takes place at $t = 0$:

$$\hat{\omega}(t) = \bar{\omega}_{\mathrm{p}} + 2\chi^0 b^\dagger b + \sum_{k=0}^{2k_0} \chi_k^0 a_k^\dagger a_k \tag{14}$$

where $\bar{\omega}_{\mathrm{p}} = \tilde{\omega}_{\mathrm{p}} - \int \mathrm{d}\omega\, k^2(\omega) P(\omega_{2k_0} - \omega)^{-1}$.

In order to eliminate the free-field operators of the external EMF and the cavity field operators at the frequency ω_{2k_0} one can make the following approximation. Inside the T-product we replace the operator $\hat{\omega}$ by a steady state value $\bar{\omega}$ in which all the number state operators are replaced by their mean values $\langle b^\dagger b\rangle$, $\langle a_k^\dagger a_k\rangle$. Neglecting the subharmonic number term $\sum_{k=0}^{2k_0} \chi_k^0 a_k^\dagger a_k$ in comparison with the quasicoherent term $2\chi^0 b^\dagger b$ we obtain for the operator $b^\dagger b$ the following equation:

$$b^\dagger b = \frac{1}{\pi} \frac{\Gamma|\beta|^2}{(\omega_{\mathrm{p}} - \bar{\omega})^2 + \Gamma^2}. \tag{15}$$

Here for frequency $\bar{\omega}$ we used its second-order approximation

$$\bar{\omega} = \bar{\omega}_{\mathrm{p}} - 2\Delta_{\mathrm{f}}$$

where $\Delta_{\mathrm{f}} = -\frac{\chi^0\Gamma|\beta|^2}{[\pi(\omega_{\mathrm{p}} - \bar{\omega}_{\mathrm{p}})^2 + \Gamma^2]}$. After introducing equation (13) into (10) and using the Born–Markov approximation we obtain the following expression:

$$\begin{aligned}
\frac{\mathrm{d}\langle \mathrm{O}(t)\rangle}{\mathrm{d}t} = {} & \mathrm{i} \sum_{k=0}^{2k_0} \left\langle \left[\bar{\omega}_k a_k^\dagger a_k, \mathrm{O}(t)\right]\right\rangle \\
& + \mathrm{i} \sum_{k_1=0}^{2k_0} \sum_{k_2=0}^{2k_0} \left\langle \left[\bar{\chi}'_{k_1,k_2} a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2} a_{2k_0-k_2} + \chi''_{k_1,k_2} a_{k_1}^\dagger a_{k_1} a_{k_2}^\dagger a_{k_2}, \mathrm{O}(t)\right]\right\rangle \\
& + \sum_{k=0}^{2k_0} \Omega_k \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{p}} t} \left\langle \left[a_k^\dagger a_{2k_0-k}^\dagger, \mathrm{O}(t)\right] - \Omega_k^* \mathrm{e}^{\mathrm{i}\omega_{\mathrm{p}} t} \left[a_k a_{2k_0-k}, \mathrm{O}(t)\right]\right\rangle \\
& + \sum_{k_1=0}^{2k_0} \sum_{k_2=0}^{2k_0} \gamma_{k_1,k_2} \left\langle \left[a_{k_1}^\dagger a_{2k_0-k_1}^\dagger \mathrm{O}(t), a_{k_2} a_{2k_0-k_2}\right] \right. \\
& \left. + \left[a_{k_2}^\dagger a_{2k_0-k_2}^\dagger, \mathrm{O}(t) a_{k_1} a_{2k_0-k_1}\right]\right\rangle
\end{aligned} \tag{16}$$

where $\bar{\omega}_k = \omega_k + \Delta_{\mathrm{f}}$, $\bar{\chi}'_{k_1,k_2} = \chi'_{k_1,k_2} - \frac{g_{k_1} g_{k_2}(\bar{\omega} - 2\omega_{\mathrm{p}})}{\hbar^2(\Gamma^2 + (\bar{\omega} - 2\omega_{\mathrm{p}})^2)}$ is the constant of interaction between the entanglement photon pairs stimulated by the second- and third-order susceptibility, $\Omega_k^2 = \frac{g_k^2}{\hbar^2 \chi_0}\Delta_{\mathrm{f}}$, Ω_k is the analogue of Rabi frequency for the excitations of photon pairs in the cavity, and $\gamma_{k_1,k_2} = \frac{g_{k_1} g_{k_2}\Gamma}{\hbar^2(\Gamma^2 + (\bar{\omega} - 2\omega_{\mathrm{p}})^2)}$ are the losses of coherent photon pairs in the cavity stimulated by the losses of the pump field inside the cavity.

As $\langle \mathrm{O}(t)\rangle = \mathrm{Tr}\{\tilde{\rho}(t)\mathrm{O}\} = \mathrm{Tr}\{\tilde{\rho}\mathrm{O}(t)\}$ in equation (16) one can pass from the Heisenberg to Schrödinger picture. After the cyclic permutation under the $\mathrm{Tr}\{\cdots\}$ operation one can replace the commutators from operator O to the density matrix of subharmonic fields $\rho(t) = \mathrm{e}^{\mathrm{i}H_0 t/\hbar} \tilde{\rho}(t) \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}$ (here $H_0 = \sum_k \hbar\bar{\omega}_k a_k^\dagger a_k$):

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\mathrm{i} \sum_{k_1=0}^{2k_0} \sum_{k_2=0}^{2k_0} [\bar{\chi}'_{k_1,k_2} a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2} a_{2k_0-k_2} + \chi''_{k_1,k_2} a_{k_1}^\dagger a_{k_1} a_{k_2}^\dagger a_{k_2}), \rho(t)]$$

$$-\sum_{k=0}^{2k_0}[\Omega \mathrm{e}^{-\mathrm{i}(\Delta_0-\Delta_\mathrm{f})t} a_k^\dagger a_{2k_0-k}^\dagger - \Omega^* \mathrm{e}^{\mathrm{i}(\Delta_0-\Delta_f)t} a_k a_{2k_0-k}, \rho(t)]$$
$$+\sum_{k_1=0}^{2k_0}\sum_{k_2=0}^{2k_0} \gamma_{k_1,k_2}\{[a_{k_1}a_{2k_0-k_1}, \rho(t)a_{k_2}^\dagger a_{2k_0-k_2}^\dagger]$$
$$+[a_{k_2}a_{2k_0-k_2}\rho(t), a_{k_1}^\dagger a_{2k_0-k_1}^\dagger]\}. \tag{17}$$

We observe that in the absorption and generation of the pump photon in the cavity, pairs of photons with the sum energy $\hbar\omega_{k_1} + \hbar\omega_{k_2} = \hbar\omega_{2k_0}$ are generated. If we decompose the density matrix of the coherent states of Boson subharmonic operators we obtain a complicated Fokker–Planck equation due to the existence of a large number of subharmonic modes in the resonance with pump fields. In the case that we have only one cavity mode in this resonance $k = k_0$, the master equation (17) is reduced to the same equation as studied in [13]. It is not difficult to observe that when the number of modes increases, the Drummond decomposition becomes difficult and for the investigation of the behaviour of photon pair generation another coherent state decomposition for the density matrix is necessary. We observe that the coefficients in the master equation (17) are smoothly dependent on the frequency of the subharmonic fields ω_k. In this situation it is convenient to replace the frequency ω_k with ω_{k_0} in all the coefficients.

In this approximation one can introduce the collective cavity field operators [5, 14]

$$I^+ = \sum_{k=0}^{2k_0} \frac{a_k^\dagger a_{2k_0-k}^\dagger}{2} \qquad I^- = \sum_{k=0}^{2k_0} \frac{a_k a_{2k_0-k}}{2} \qquad I_z = \sum_{k=0}^{2k_0} \tfrac{1}{2}\left(a_k^\dagger a_k + \tfrac{1}{2}\right) \tag{18}$$

which satisfy the following commutators for the operators of $SU(1, 1)$ algebra:

$$[I^+, I^-] = -2I_z \qquad [I_z, I^\pm] = \pm I^\pm. \tag{19}$$

Thus, the density matrix equation $W(t) = \exp\{\mathrm{i}(\omega_\mathrm{p} - \bar{\omega}_{2k_0})I_z\}\rho(t)\exp\{-\mathrm{i}(\omega_\mathrm{p} - \bar{\omega}_{2k_0})I_z\}$ can be represented in the following form:

$$\frac{\partial W(t)}{\partial t} = -\mathrm{i}[\chi I^+I^- + \Delta I_z + \mathrm{i}\{\Omega^* I^- - \Omega I^+\}, W(t)] + \gamma\{[I^-W(t), I^+] + [I^-, W(t)I^+]\} \tag{20}$$

where

$$\chi = 4(\bar{\chi}'_{k_0,k_0} - \chi''_{k_0,k_0}) \qquad \Delta = \Delta_0 - \Delta_\mathrm{f} + \chi''_{k_0,k_0}$$
$$\Omega = 2\Omega_{k_0} \qquad \gamma = 4\gamma_{k_0,k_0}.$$

It is not difficult to observe that the Casimir operator

$$I^2 = (I_z)^2 - 1/2(I^+I^- + I^-I^+) \tag{21}$$

which satisfies the commutator relations $[I^2, I^\pm] = [I^2, I_z] = 0$ is conserved. The discrete representation of $SU(1, 1)$ Lie algebra is described by the state vectors $|m, j\rangle$ that satisfy [10, 15]

$$\begin{aligned} I^2|m, j\rangle &= j(j-1)|m, j\rangle \\ I_z|m, j\rangle &= (m+j)|m, j\rangle \\ I^+|m, j\rangle &= \sqrt{(m+1)(m+2j)}|m+1, j\rangle \\ I^-|m, j\rangle &= \sqrt{m(m+2j-1)}|m-1, j\rangle \end{aligned} \tag{22}$$

where $I^-|0, j\rangle = 0$. Here j is the Bargmann index and m is any non-negative integer. The set $|m, j\rangle$, $(m = 0, 1, 2, \ldots; j = \text{const})$ becomes the complete orthonormal basis

$$\begin{aligned} \langle j, m|n, j\rangle &= \delta_{m,n} \\ \sum_{m=0}^{\infty} |m, j\rangle\langle j, m| &= 1. \end{aligned} \tag{23}$$

In analogy with the Dicke co-operating number $j = N/2$ for $SU(2)$ algebra one can introduce the co-operative number j for distinguishing the conjugate mode pairs $2k_0 - k_i, k_i$, $i = 1, 2, \ldots, N$. Using the conservation vector $I^2 = j(j-1)$ one can derive that the co-operative number for the pairs of photons is $j = \sum_{k=0}^{2k_0} 1/4 = N/2$.

In the next section the stationary solution for master equation (20) will be analysed. This solution allows us to obtain mean values for the number of pairs of entanglement photons $\langle I^+ I^- \rangle$, number of photons $\langle I_z \rangle$ and their fluctuations $\delta^2 = \langle I_z^2 \rangle - \langle I_z \rangle^2$.

3. Fokker–Planck equation and its steady state solution

Following the decomposition of the density matrix on non-diagonal generalized P representation for Bose algebra [13] one can introduce the following decomposition on coherent states for $SU(1, 1)$ algebra:

$$W = \int_D P(\alpha, \beta) \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle} \, \mathrm{d}\mu\,(\alpha, \beta). \tag{24}$$

Here D is the integration domain, $\mathrm{d}\mu\,(\alpha, \beta) = \mathrm{d}\alpha\,\mathrm{d}\beta$ is the integration measure,

$$|\alpha\rangle = (1 - |\alpha|^2)^j \exp(\alpha I^\dagger)|0, j\rangle$$
$$\langle\beta^*| = (1 - |\beta|^2)^j \langle j, 0| \exp(\beta I^-)$$

are the coherent states for the $SU(1, 1)$ algebra and

$$\langle\beta^*|\alpha\rangle = \frac{(1 - |\alpha|^2)^j (1 - |\beta|^2)^j}{(1 - \alpha\beta)^{2j}}$$

is the normalization coefficient for the projector operator $|\alpha\rangle\langle\beta^*|$. Using the following action of operators I^+, I^-, I_z of $SU(1, 1)$ algebra on the coherent state:

$$I^+|\alpha\rangle = \left(1 - |\alpha|^2\right)^j \frac{\partial}{\partial\alpha} \exp\left(\alpha I^+\right)|0, j\rangle$$
$$I^-|\alpha\rangle = (1 - |\alpha|^2)^j \left(\alpha^2 \frac{\partial}{\partial\alpha} + 2j\alpha\right) \exp\,(\alpha I^+)|0, j\rangle$$
$$I_z|\alpha\rangle = (1 - |\alpha|^2)^j \left(\alpha \frac{\partial}{\partial\alpha} + j\right) \exp\,(\alpha I^+)|0, j\rangle$$

one can obtain the following Fokker–Planck equation:

$$\begin{aligned} \frac{\partial}{\partial t} P(\alpha, \beta) = {} & \frac{\partial}{\partial\alpha} \left(2\mathrm{i}j\chi\alpha \frac{1 + \alpha\beta}{1 - \alpha\beta} + \mathrm{i}\Delta\alpha - \Omega(\alpha^2 - 1) + 2j\gamma\alpha\right) P(\alpha, \beta) \\ & + \frac{\partial}{\partial\beta} \left(-2\mathrm{i}j\chi\beta \frac{1 + \alpha\beta}{1 - \alpha\beta} - \mathrm{i}\Delta\beta - \Omega(\beta^2 - 1) + 2j\gamma\beta\right) P(\alpha, \beta) \\ & - \frac{\partial^2}{\partial\alpha^2} (\gamma + \mathrm{i}\chi)\alpha^2 P(\alpha, \beta) - \frac{\partial^2}{\partial\beta^2} (\gamma - \mathrm{i}\chi)\beta^2 P(\alpha, \beta) \\ & + 2\gamma \frac{\partial^2}{\partial\alpha\partial\beta} \alpha^2 \beta^2 P(\alpha, \beta). \end{aligned} \tag{25}$$

For many problems in quantum optics it is sufficient to know the steady state solution of the Fokker–Planck equation. Representing the steady state solution in the potential form

$$P(\alpha, \beta) = N \exp(-\Phi(\alpha, \beta)) \tag{26}$$

one can obtain the following differential equations for the potential $\Phi(\alpha, \beta)$:

$$
\begin{aligned}
&(\gamma + \mathrm{i}\chi)\alpha^2 \frac{\partial \Phi}{\partial \alpha} - \gamma\alpha^2\beta^2 \frac{\partial \Phi}{\partial \beta} \\
&\qquad = 2\mathrm{i}j\chi\alpha \frac{1+\alpha\beta}{1-\alpha\beta} - \mathrm{i}\Delta\alpha + \Omega(\alpha^2 - 1) - 2j\gamma\alpha + 2(\gamma + \mathrm{i}\chi)\alpha - 2\gamma\alpha^2\beta \\
&-\gamma\alpha^2\beta^2 \frac{\partial \Phi}{\partial \alpha} + (\gamma - \mathrm{i}\chi)\beta^2 \frac{\partial \Phi}{\partial \beta} \\
&\qquad = 2\mathrm{i}j\chi\beta \frac{1+\alpha\beta}{1-\alpha\beta} - \mathrm{i}\Delta\beta + \Omega(\beta^2 - 1) - 2j\gamma\beta + 2(\gamma - \mathrm{i}\chi)\beta - 2\gamma\alpha\beta^2.
\end{aligned} \tag{27}
$$

We observe that for the arbitrary parameters Δ and χ the so-called potential condition for $\Phi(\alpha, \beta)$ [6]

$$
\frac{\partial^2 \Phi(\alpha, \beta)}{\partial\beta\partial\alpha} = \frac{\partial^2 \Phi(\alpha, \beta)}{\partial\alpha\partial\beta} \tag{28}
$$

is not satisfied.

For solving equation (20) we can examine the case when $\chi = \Delta = 0$. It is not difficult to observe that in this case the potential condition (28) is satisfied and the steady state solution can be written as

$$
P(\alpha, \beta) = \frac{1}{N(y)} \frac{1}{\alpha^2\beta^2} \left(\frac{1}{\alpha\beta} - 1\right)^{-2j} \exp\left[-\frac{\Omega}{\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right]. \tag{29}
$$

The normalization constant is given by the relation

$$
N(y) = -\frac{4\pi^2}{\Gamma(2j)} y^{2j-1} I_{2j-1}(2y) \tag{30}
$$

where $y^2 = \frac{\Omega^2}{\gamma^2}$ and $I_\nu(z)$ represents the traditional Bessel function.

Now we consider the situation when $\chi \neq 0$ and the detuning $\Delta = 0$. In this case the potential condition (28) remains unsatisfied, but it can be satisfied if one introduces the two terms $-\mathrm{i}\chi\partial^2/(\partial\alpha\partial\beta)[\alpha^2\beta^2 P(\alpha, \beta)]$ and $+\mathrm{i}\chi\partial^2/(\partial\beta\partial\alpha)[\alpha^2\beta^2 P(\alpha, \beta)]$ in equation (25). As in deriving the Fokker–Planck equation we have considered that $\partial^2/(\partial\beta\partial\alpha)P(\alpha, \beta) = \partial^2/(\partial\alpha\partial\beta)P(\alpha, \beta)$, then these two terms in the right-hand side of Fokker–Planck equation (25) give zero contribution. After this, equation (25) suffers some modification. The steady state solution of the Fokker–Planck equation can be obtained from the equations

$$
\begin{aligned}
0 &= \bigg(-2\mathrm{i}j\chi\alpha \frac{1+\alpha\beta}{1-\alpha\beta} - \Omega(\alpha^2 - 1) + 2j\gamma\alpha - \frac{\partial}{\partial\alpha}(\gamma + \mathrm{i}\chi)\alpha^2 \\
&\qquad + (\gamma - \mathrm{i}\chi)\frac{\partial}{\partial\beta}\alpha^2\beta^2\bigg) P(\alpha, \beta) \\
0 &= \bigg(2\mathrm{i}j\chi\beta \frac{1+\alpha\beta}{1-\alpha\beta} - \Omega(\beta^2 - 1) + 2j\gamma\beta - \frac{\partial}{\partial\beta}(\gamma - \mathrm{i}\chi)\beta^2 \\
&\qquad + (\gamma + \mathrm{i}\chi)\frac{\partial}{\partial\alpha}\alpha^2\beta^2\bigg) P(\alpha, \beta).
\end{aligned} \tag{31}
$$

Using the representation of P function through potential $\Phi(\alpha, \beta)$ from equations (31) one can obtain the following differential equations which satisfy the potential condition (28):

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \alpha} &= \frac{2j}{\alpha(\alpha\beta - 1)} - \frac{\Omega}{\alpha^2(\gamma + \mathrm{i}\chi)} + \frac{2}{\alpha} \\
\frac{\partial \Phi}{\partial \beta} &= \frac{2j}{\beta(\alpha\beta - 1)} - \frac{\Omega}{\beta^2(\gamma - \mathrm{i}\chi)} + \frac{2}{\beta}.
\end{aligned} \tag{32}
$$

After introducing the potential $\Phi(\alpha,\beta)$ determined from equations (32) in relation (26) we obtain the following relation from P function:

$$P(\alpha,\beta)=\frac{1}{N(y_1)}\frac{1}{\alpha^2\beta^2}\left(\frac{1}{\alpha\beta}-1\right)^{-2j}\exp\left[-\Omega\left(\frac{1}{\alpha(\gamma-\mathrm{i}\chi)}+\frac{1}{\beta(\gamma+\mathrm{i}\chi)}\right)\right]. \tag{33}$$

Here the normalization constant depends on the $y_1^2=\frac{\Omega^2}{\gamma^2+\chi^2}$.

In this situation, from definition (24) one can obtain the following expression for the mean value of the physical quantity O:

$$\langle \mathrm{O}\rangle=\sum_{n=0}^{\infty}\int_D P(\alpha,\beta)\frac{\langle n|\alpha\rangle\langle\beta^*|\mathrm{O}|n\rangle}{\langle\beta^*|\alpha\rangle}\,\mathrm{d}\mu\,(\alpha,\beta). \tag{34}$$

Using the solution of Fokker–Planck equation (33) we represent below the analytical dependence of mean values of the following operators $\mathrm{O}=(I_z,I_z^2,I^+I^-)$:

$$\begin{aligned}\langle I_z\rangle&=j+y_1\frac{I_{2j}(2y_1)}{I_{2j-1}(2y_1)}\\ \langle I_z^2\rangle&=j^2+(2j+1)y_1\frac{I_{2j}(2y_1)}{I_{2j-1}(2y_1)}+y_1^2\frac{I_{2j+1}(2y_1)}{I_{2j-1}(2y_1)}\\ \langle I^+I^-\rangle&=2jy_1\frac{I_{2j}(2y_1)}{I_{2j-1}(2y_1)}+y_1^2\frac{I_{2j+1}(2y_1)}{I_{2j-1}(2y_1)}\end{aligned} \tag{35}$$

and

$$\delta^2=y_1\frac{I_{2j}(2y_1)}{I_{2j-1}(2y_1)}+y_1^2\left(\frac{I_{2j+1}(2y_1)}{I_{2j-1}(2y_1)}-\frac{I_{2j}^2(2y_1)}{I_{2j-1}^2(2y_1)}\right). \tag{36}$$

Let now study the behaviour of the generation of entanglement photon pairs $\langle I_z\rangle-j$ for small values for argument $y_1\ll 1$. In this case the Bessel function is approximated by $I_n(2x)\approx\frac{x^n}{\Gamma(n+1)}$ and

$$\langle I_z\rangle\approx j+\frac{y_1^2}{2j-1}.$$

The relative fluctuations of the number of photon pairs σ as a function at the intensity of the external EMF is

$$\sigma=\frac{\sqrt{\langle I_z^2\rangle-\langle I_z\rangle^2}}{\langle I_z\rangle-j}\approx\frac{\sqrt{2j-1}}{y_1}.$$

For large argument limit of the Bessel function the behaviour of the mean value $\langle I_z\rangle$ and relative fluctuations of number of photon pairs σ are

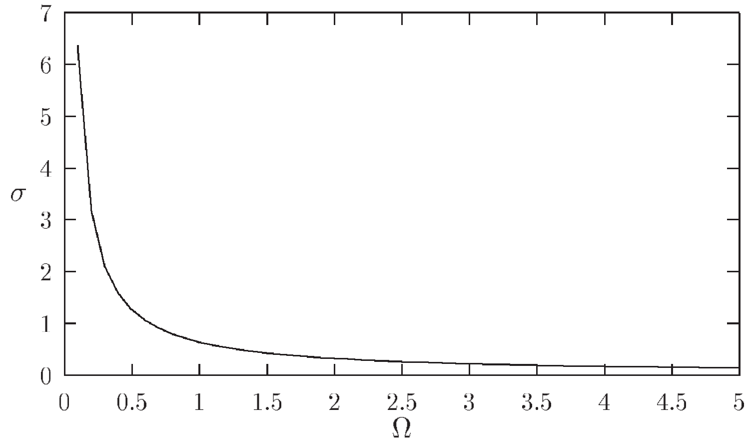
$$\langle I_z\rangle\approx j+y_1$$

and

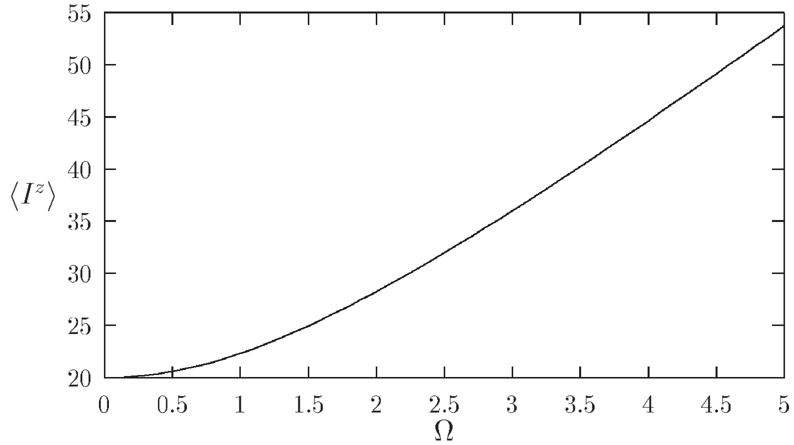
$$\sigma\approx\frac{1}{\sqrt{y_1}}.$$

By increasing the external pump field the relative fluctuation δ passes from $1/y_1$ to $1/\sqrt{y_1}$ dependence on y_1. The enhancement of the number of photon pairs as a function of the external field is observed from this expressions too (the numerical results are plotted in figure 1)

In this section we observe that the steady state solution is difficult to obtain in the case of $\Delta\neq 0$. In the next section we propose a method of representing the density matrix through the antinormal product of operators $I^+$and I^-. This method makes it possible to solve the stationary master equation for $\Delta\neq 0$.



(*a*)



(*b*)

Figure 1. The dependences of the relative fluctuations of number of photon pairs σ (*a*) and the number of biphotons $\langle I^z \rangle$ (*b*) as function of frequency Ω for $j = 20$, $\gamma = 0.01$, $\chi = 0.1$ and $\Delta = 0$.

4. The antinormal representation of the steady state solution of the master equation

In order to obtain the solution of master equation (20) for arbitrary detuning and arbitrary third-order nonlinearity we represent the density matrix of the steady state master equation (20)

$$\mathrm{i}[\chi I^+ I^- + \Delta I_z + \mathrm{i}\{\Omega^* I^- - \Omega I^+\}, W_{\mathrm{s}}] - \gamma\{[I^- W_{\mathrm{s}}, I^+] + [I^-, W_{\mathrm{s}} I^+]\} = 0 \tag{37}$$

through antinormal ordering operators I^+ and I^-. The same representation was used in the papers [8] for $SU(2)$ algebra. Here we extend this method for $SU(1, 1)$ symmetry. Following the elegant method developed in [8] we are looking for a solution of equation (35) of the form

$$W_{\mathrm{s}} = A^{-1} F(I^-) F^+(I^+) \tag{38}$$

where $A = \mathrm{Tr}[F(I^-)F^+(I^+)]$, $F(I^-)$ and $F^+(I^+)$ are operator functions of I^- and I^+, respectively. Here the function $F(I^\pm)$ can be represented in a Taylor series

$$F(I^\pm) = \sum_{n=0}^{\infty} C_n \left(I^\pm\right)^n. \tag{39}$$

By using the commutation rules corresponding to $SU(1,1)$ symmetry, it is easy to demonstrate the following operator identities:

$$I_z F(I^-) = -F(I^-)I_z - \left[I^+, \int F(I^-)\,\mathrm{d}I^-\right] \tag{40}$$

$$[I^+I^-, F(I^-)F(I^+)] = [I^+, I^-F(I^-)]F(I^+) - \text{H.c.} \tag{41}$$

where H.c. stands for the Hermitian conjugate and

$$\int F(I^-)\,\mathrm{d}I^- = \sum_{n=0}^{\infty} \frac{C_n}{n+1}\left(I^-\right)^{n+1}.$$

The operator equation (37) can be represented in the form

$$\left[I^+, G(I^-)\right]F^+(I^+) + \text{H.c.} = 0 \tag{42}$$

where

$$G(I^-) = I^-F(I^-)\left(-\mathrm{i}\frac{\chi}{\gamma} - 1\right) - \mathrm{i}\frac{\Delta}{\gamma}\int F(I^-)\,\mathrm{d}I^- - \frac{\Omega}{\gamma}F(I^-). \tag{43}$$

From equation (42) it follows that the commutator $[I^+, G(I^-)]$ must be proportional to $F(I^-)$, but in view of the commutation relations (19) this is not possible. Thus, in order to satisfy equation (42) it is necessary that the commutator $[I^+, G(I^-)]$ be zero. This is possible when $G(I^-) = \text{const}$, that is

$$I^-F(I^-)\left(1 + \mathrm{i}\frac{\chi}{\gamma}\right) - \mathrm{i}\frac{\Delta}{\gamma}\int F(I^-)\,\mathrm{d}I^- + \frac{\Omega}{\gamma}F(I^-) = \text{const}. \tag{44}$$

The solution of equation (44) can be written in a compact form:

$$F(I^-) = c\left(I^- - \mathrm{i}d\right)^{-(1+\xi)} \tag{45}$$

where $d = \frac{\mathrm{i}\Omega}{\gamma+\mathrm{i}\chi}$ and $\xi = -\frac{\mathrm{i}\Delta}{\gamma+\mathrm{i}\chi}$. Finally, the stationary density matrix can be represented in the form

$$\begin{aligned} W_\mathrm{s} &= |c|^2(I^- - \mathrm{i}d)^{-(1+\xi)}\left(I^+ + \mathrm{i}d^*\right)^{-(1+\xi^*)} \\ &= \lim_{n_0\to\infty} D^{-1}\sum_{k,l}^{n_0} \mathrm{i}^{k-l}d^{-k}(d^*)^l \frac{\Gamma(1+\xi+k)\Gamma(1+\xi^*+l)}{k!l!\Gamma(1+\xi)\Gamma(1+\xi^*)}(I^-)^k(I^+)^l \end{aligned} \tag{46}$$

where D is the normalization factor so that $\mathrm{Tr}\{W_\mathrm{s}\} = 1$, and $\Gamma(z)$ is the Γ-function. The normalization constant D is given by the limit $D = \lim_{n_0\to\infty} D(n_0)$, where

$$D(n_0) = \sum_{l=0}^{n_0} |d|^{-2l}\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)}\sum_{p=j}^{n_0}\frac{\Gamma(l+p+j)\Gamma(l+p-j+1)}{\Gamma(p+j)\Gamma(p-j+1)}. \tag{47}$$

In analogy with the Fokker–Planck method one can obtain the following values for the mean number of operators I_z, $(I_z)^2$ and I^+I^-:

$$\langle \mathrm{O}\rangle = \lim_{n_0\to\infty}\frac{1}{D}\sum_{l=0}^{n_0}|d|^{-2l}\,\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)}\sum_{p=j}^{n_0} f_\mathrm{p}(\mathrm{O})\frac{\Gamma(l+p+j)\Gamma(l+p-j+1)}{\Gamma(p+j)\Gamma(p-j+1)}. \tag{48}$$

Here

$$f_\mathrm{p}(\mathrm{O}) = \begin{cases} p & \text{if} \quad \mathrm{O} \equiv I_z \\ p^2 & \text{if} \quad \mathrm{O} \equiv I_z^2 \\ p(p+1) - j(j+1) & \text{if} \quad \mathrm{O} \equiv I^+I^-. \end{cases}$$

We observed that this method gives results which slightly differ from expression (35). The difference consists in the representation of the mean value through two sums in expression (48). In the next section this difference is analysed.

5. Results and discussions

Let us consider the first case, when the detuning $\Delta = 0$. This situation corresponds to strong resonance between the external pump coherent field and cavity mode $2k_0$. In order to neglect the field detuning $\Delta_{\rm f}$ as compared to parameter χ in master equation (20) one supposes that the third-order susceptibility at frequency $\omega_{\rm p}$ is less than the same susceptibility at frequency $\omega_{k_0}[\chi^{(3)}(\omega_{\rm p}, \omega_{k_0}) \ll \chi^{(3)}(\omega_{k_0}, \omega_{k_0})]$ and that the value of the intensity of the external pump field does not affect the inequality $\Delta_{\rm f} \ll \chi$. In this case one can use the solution obtained by the Fokker–Planck method.

From the Fokker–Planck and antinormal ordering methods it follows that the expressions for the mean values of the physical quantities slightly differ. We observe that Fokker–Planck methods do not allow us to find the steady state solution for the arbitrary detuning Δ and χ. The antinormal ordering method allows us to do this, but it expresses the mean value for operator O through the ratio of two double-divergent sums and it is difficult to do a numerical simulation of expression (48). It is interesting to find the approximate mathematical connections between expressions (48) and (35). For this we do some mathematical transformation of expression (48) in the case $\Delta = 0$. We change the sum of the variables in expression (48) $n = p - j$ and $m = l + n$ in order to obtain the following expression for $\langle \mathrm{O} \rangle$:

$$\langle \mathrm{O} \rangle = \frac{\sum_{n=0}^{\infty} |d|^{2n} f_n(\mathrm{O}) \, (n!\Gamma(2j+n))^{-1} \sum_{m=n}^{\infty} |d|^{-2m} \, \Gamma(m+2j)\Gamma(m+1)}{\sum_{n=0}^{\infty} |d|^{2n} \, (n!\Gamma(2j+n))^{-1} \sum_{m=n}^{\infty} |d|^{-2m} \, \Gamma(m+2j)\Gamma(m+1)}. \tag{49}$$

This expression is more similar to expression (35), but under the sum on n we have the divergent sum $\sum_{m=n}^{\infty} |d|^{-2m} \, \Gamma(m+2j)\Gamma(m+1)$. If we change the summations in (49) to the integer parameter n_0 in the limit, when $n_0 \to \infty$, one can multiply the numerator and denominator of this expression by $|d|^{2n_0}$. Making the change of variable $p = n_0 - m$ we obtain the following formula:

$$\langle \mathrm{O} \rangle = \lim_{n_0 \to \infty} \frac{\sum_{n=0}^{n_0} |d|^{2n} f_n(R) \, (n!\Gamma(2j+n))^{-1} \, S(n, n_0)}{\sum_{n=0}^{n_0} |d|^{2n} \, (n!\Gamma(2j+n))^{-1} \, S(n, n_0)} \tag{50}$$

where $S(n, n_0) = \sum_{p=0}^{n_0-n} |d|^{2p} \, \Gamma(n_0 - p + 2j)\Gamma(n_0 - p + 1)$. We observe that for $n \ll n_0$ the sum $S(n, n_0)$ slowly depends on parameter n and in expression (50) one can simplify the numerator and the denominator by $S(n_0)$. Under this supposition equations (35) and (49) coincide.

Let us now discuss the behaviour of the cavity subharmonic EMF, when the detuning Δ is different from zero. In order to obtain the convergent sums in (48) we divide the numerator and denominator $D(n_0)$ in (48) of expression $(n_0)^2$. In this case one obtains the convergent expressions of the numerator and denominator.

The main interesting effect in this case is described by the dependence of Δ on the input pumping coherent EMF:

$$\Delta = \Delta_0 + \Delta_{\rm f}$$

where $\Delta_0 \approx \omega_{\rm p} - \omega_{2k_0}$ is the part of detuning which does not depend on the intensity of the external EMF and $\Delta_{\rm f} = \Gamma\chi^0\Omega^2/[\gamma(\Gamma^2 + (\bar{\omega} - 2\omega_{\rm p})^2)]^{-1}$ is the detuning part, which is proportional to the intensity of the external coherent field. If the sign of detuning Δ_0 is opposite to the field-dependent detuning $\Delta_{\rm f}$ in the process of increasing the external pump EMF these two detunings give zero value for the summed detuning Δ. At this point the enhancement of the generation rate of biphotons takes place.

Returning to the definition of $\xi = -\frac{\mathrm{i}\Delta}{\gamma + \mathrm{i}\chi}$, in the case $\Delta_{\rm f} = 0$ one observes that the product

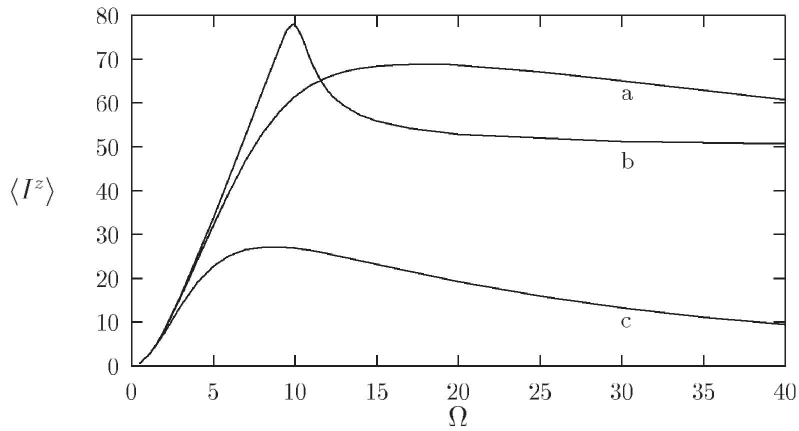


Figure 2. The dependence of the number of biphotons $\langle I^z \rangle$ as function of frequency Ω for $n_0 = 100$, $j = 20$, $\Delta_0 = 0.01$, $\gamma = 0.01$, $\chi = 0.1$ and (a) $\Delta_f = 0.01\Omega^2$, (b) $\Delta_f = 0$, (c) $\Delta_f = 0.1\Omega^2$.

of the expression

$$\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)} = \prod_{k=0}^{l}[(\mathrm{Re}\,\xi + k)^2 + (\mathrm{Im}\,\xi)^2] \tag{51}$$

for $\gamma \ll \chi$, which corresponds to $\mathrm{Im}\,\xi \ll \mathrm{Re}\,\xi$, will be transformed into $\prod_{k=0}^{l}(\mathrm{Re}\,\xi + k)^2$. On the other hand, the sums on l in (47) and (48) become truncated for $l^* = \Delta/\chi$. The infinity series becomes finite. It is clear that with increasing field strength the number of generated biphotons tends to the constant value due to the fact that the expression for the mean number is obtained from the ratio of two power polynomials of the EMF strength. In figure 2 we represent some numerical simulations of dependence I^z as a function of Rabi frequency for different values of the field detuning. We observe that with the increase of the field detuning for large strength of the EMF the mean number of generated photons in the cavity tends to zero.

The steady state solutions of the master equation obtained by the two proposed methods differ even in the absence of the external EMF:

$$\mathrm{i}\chi[I^+I^-, W(t)] - \gamma\{[I^-W(t), I^+] + [I^-, W(t)I^+]\} = 0. \tag{52}$$

It is not difficult to observe that from the Fokker–Planck method one obtains the ground state solution $W_s = |0, j\rangle\langle j, 0|$ while the antinormal method gives the solution $W_s = D\left(I^+I^-\right)^{-1}$ $(W_s = D\sum_{m=0}^{\infty}[(m+1)(m+2j)]^{-1}|m+1, j\rangle\langle j, m+1|)$. The second solution represents the nonradiant excited quantum state of the system. The linear combination of these two solutions is also the solution of equation (52). In order to obtain the ground state solution by the antinormal method in the absence of the external field it is required that the steady state solution of equation (52) be the ground state solution $|0, j\rangle\langle j, 0|$. From this requirement the normalization constant D must be represented through the sum of the inverse value of the strength of the external pump field $1/d$, so that the ratio of two divergent series in (46) (when the intensity of the external field tends to zero) gives the ground state. This condition is necessary to obtain further physical solutions for master equations.

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